

Braided m-Lie Algebras

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Abstract

Braided m-Lie algebras induced by multiplication are introduced, which generalize Lie algebras, Lie color algebras and quantum Lie algebras. The necessary and sufficient conditions for the braided m-Lie algebras to be strict Jacobi braided Lie algebras are given. Two classes of braided m-Lie algebras are given, which are generalized matrix braided m-Lie algebras and braided m-Lie subalgebras of $End_F M$, where M is a Yetter-Drinfeld module over B with $\dim B < \infty$. In particular, generalized classical braided m-Lie algebras $sl_{q,f}(GM_G(A), F)$ and $osp_{q,t}(GM_G(A), M, F)$ of generalized matrix algebra $GM_G(A)$ are constructed and their connection with special generalized matrix Lie superalgebra $sl_{s,f}(GM_{\mathbf{Z}_2}(A^s), F)$ and orthosymplectic generalized matrix Lie super algebra $osp_{s,t}(GM_{\mathbf{Z}_2}(A^s), M^s, F)$ are established. The relationship between representations of braided m-Lie algebras and their associated algebras are established.

0 Introduction

The theory of Lie superalgebras has been developed systematically, which includes the representation theory and classifications of simple Lie superalgebras and their varieties [8] [4]. In many physical applications or in pure mathematical interest, one has to consider not only \mathbf{Z}_2 - or \mathbf{Z} - grading but also G -grading of Lie algebras, where G is an abelian group equipped with a skew symmetric bilinear form given by a 2-cocycle. Lie algebras in symmetric and more general categories were discussed in [7] and [6]. A sophisticated multilinear version of the Lie bracket was considered in [9] [12]. Various generalized Lie algebras have already appeared under different names, e.g. Lie color algebras, ϵ Lie

algebras [13], quantum and braided Lie algebras, generalized Lie algebras [2] and H -Lie algebras [3].

In [10], Majid introduced braided Lie algebras from geometrical point of view, which have attracted attention in mathematics and mathematical physics (see e.g. [11] and references therein).

In this paper we introduce braided m-Lie algebras and (strict) Jacobi braided Lie algebras from the algebraic point of view, rather than geometrical view point as in [10]. Our braided m-Lie algebras are different from the braided Lie algebras defined by Majid [10]. This is verified by giving an example of braided m-Lie algebras which is neither a braided Lie algebra (in the sense of Majid) nor an Jacobi braided Lie algebra. We give the necessary and sufficient conditions for the braided m-Lie algebras to be strict Jacobi braided Lie algebras. In section 2 we give G -gradings of generalized matrix algebras and construct braided m-Lie algebras corresponding to such algebras, which are called generalized matrix braided m-Lie algebras. This leads to the braided m-Lie algebra construction on path algebras, full matrix algebras and simple algebras. We construct generalized classical braided m-Lie algebras of generalized matrix algebras. In particular, special generalized matrix Lie color algebra $sl_{q,f}(GM_G(A), F)$ and ortho-symplectic generalized matrix Lie color algebra $osp_{q,t}(GM_G(A), M, F)$ are related to the corresponding Lie superalgebras. In Section 3 we give another class of braided m-Lie algebras, i.e. braided m-Lie subalgebras of $(End_F M)^-$. We show that representations of an algebra A associated to a braided m-Lie algebra L are also representations of L . Furthermore, we show that if (M, ψ) is a faithful representation of L , then representations of $End_F M$ are also ones of L .

Throughout, F is a field, G is an additive group and r is a bicharacter of G ; $|x|$ denotes the degree of x and (\mathcal{C}, C) is a braided tensor category with braiding C . We write $W \otimes f$ for $id_W \otimes f$ and $f \otimes W$ for $f \otimes id_W$. Algebras discussed here may not have unity element.

1 Braided m-Lie Algebras

In this section we introduce braided m-Lie algebras and (strict) Jacobi braided Lie algebras. We give the necessary and sufficient conditions for the braided m-Lie algebras to be strict Jacobi braided Lie algebras.

Definition 1.1 *Let $(L, [\])$ be an object in the braided tensor category (\mathcal{C}, C) with morphism $[\] : L \otimes L \rightarrow L$. If there exists an algebra (A, m) in (\mathcal{C}, C) and monomorphism $\phi : L \rightarrow A$ such that $\phi[\] = m(\phi \otimes \phi) - m(\phi \otimes \phi)C_{L,L}$, then $(L, [\])$ is called a braided m-Lie algebra in (\mathcal{C}, C) induced by multiplication of A through ϕ . Algebra (A, m) is called an algebra associated to $(L, [\])$.*

A Lie algebra is a braided m -Lie algebra in the category of ordinary vector spaces, a Lie color algebra is a braided m -Lie algebra in symmetric braided tensor category (\mathcal{M}^{FG}, C^r) since the canonical map $\sigma : L \rightarrow U(L)$ is injective (see [13, Proposition 4.1]), a quantum Lie algebra is a braided m -Lie algebra in the Yetter-Drinfeld category $({}^B_B\mathcal{YD}, C)$ by [5, Definition 2.1 and Lemma 2.2]), and a “good” braided Lie algebra is a braided m -Lie algebra in the Yetter-Drinfeld category $({}^B_B\mathcal{YD}, C)$ by [5, Definition 3.6 and Lemma 3.7]). For a cotriangular Hopf algebra (H, r) , the (H, r) -Lie algebra defined in [3, 4.1] is a braided m -Lie algebra in the braided tensor category $({}^H\mathcal{M}, C^r)$. Therefore, the braided m -Lie algebras generalize most known generalized Lie algebras.

For an algebra (A, m) in (\mathcal{C}, C) , obviously $L = A$ is a braided m -Lie algebra under operation $[\] = m - mC_{L,L}$, which is induced by A through id_A . This braided m -Lie algebra is written as A^- .

If V is an object in \mathcal{C} and $C_{V,V} = C_{V,V}^{-1}$, then we say that the braiding is symmetric on V .

Example 1.2 *If H is a braided Hopf algebra in the Yetter-Drinfeld module category $({}^B_B\mathcal{YD}, C)$ with $B = FG$ and $C(x \otimes y) = r(|y|, |x|)y \otimes x$ for any homogeneous elements $x, y \in H$, then $P(H) =: \{x \in H \mid x \text{ is a primitive element}\}$ is a braided m -Lie algebra iff the braiding C is symmetric on $P(H)$.*

Indeed, it is easy to check that $P(H)$ is the Yetter-Drinfeld module. By simple computation we have $\Delta([x, y]) = [x, y] \otimes 1 + 1 \otimes [x, y] + (1 - r(|x|, |y|)r(|y|, |x|))x \otimes y$ for any homogeneous elements $x, y \in P(H)$. Thus $[x, y] \in P(H)$ iff $r(|x|, |y|)r(|y|, |x|) = 1$, as asserted.

Theorem 1.3 *Let $(L, [\])$ be a braided m -Lie algebra in (\mathcal{C}, C) .*

(i) $(L, [\])$ satisfies the braided anti-symmetry (or quantum anti-symmetry):

(BAS): $[\] = -[\]C_{L,L}$

if and only if $mC_{L,L} = mC_{L,L}^{-1}$.

(ii) If the braided anti-symmetry holds, then braided m -Lie algebra $(L, [\], m)$ satisfies the (left) braided primitive Jacobi identity:

(BJ): $[\](L \otimes [\]) + [\](L \otimes [\])(L \otimes C_{L,L}^{-1})(C_{L,L} \otimes L) + [\](L \otimes [\])(C_{L,L}^{-1} \otimes L)(L \otimes C_{L,L}) = 0$,

and the right braided primitive Jacobi identity:

(BJI'): $[\]([\] \otimes L) + [\]([\] \otimes L)(L \otimes C_{L,L}^{-1})(C_{L,L} \otimes L) + [\]([\] \otimes L)(C_{L,L}^{-1} \otimes L)(L \otimes C_{L,L}) = 0$.

Proof . (i) Assume that $(L, [\])$ satisfies the braided anti-symmetry. Since $[\] = -[\]C$ we have $m - mC = mCC - mC$ and $m = mCC$, which implies $mC = mC^{-1}$. The necessity is clear.

(ii)

$$\begin{aligned}
\text{l.h.s. of (BJI)} &= m(L \otimes m) - m(L \otimes m)(L \otimes C) \\
&\quad - mC(L \otimes m) + mC(L \otimes m)(L \otimes C) \\
&\quad + m(L \otimes m)(L \otimes C^{-1})(C \otimes L) \\
&\quad + m(L \otimes m)(L \otimes C)(L \otimes C^{-1})(C \otimes L) \\
&\quad - mC(L \otimes m)(L \otimes C^{-1})(C \otimes L) \\
&\quad + mC(L \otimes m)(L \otimes C)(L \otimes C^{-1})(C \otimes L) \\
&\quad + m(L \otimes m)(C^{-1} \otimes L)(L \otimes C) \\
&\quad - m(L \otimes m)(L \otimes C)(C^{-1} \otimes L)(L \otimes C) \\
&\quad - mC(L \otimes m)(C^{-1} \otimes L)(L \otimes C) \\
&\quad + mC(L \otimes m)(L \otimes C)(C^{-1} \otimes L)(L \otimes C).
\end{aligned}$$

We first check that $-(6\text{th term}) = 12\text{th term}$ and $4\text{th term} = -(10\text{th term})$. Indeed,

$$\begin{aligned}
12\text{th term} &= mC^{-1}(L \otimes m)(L \otimes C^{-1})(C^{-1} \otimes L)(L \otimes C) \\
&= m(m \otimes L)(L \otimes C^{-1})(C^{-1} \otimes L)(L \otimes C^{-1})(C^{-1} \otimes L)(L \otimes C) \\
&= m(L \otimes m)(L \otimes C)(C^{-1} \otimes L)(L \otimes C^{-1})(C^{-1} \otimes L)(L \otimes C) \\
&= m(L \otimes m)(C^{-1} \otimes L)(L \otimes C^{-1})(C \otimes L)(C^{-1} \otimes L)(L \otimes C) \\
&= -(6\text{th term}).
\end{aligned}$$

$$\begin{aligned}
4\text{th term} &= m(m \otimes L)(C^{-1} \otimes L)(L \otimes C^{-1})(C^{-1} \otimes L) \\
&= m(L \otimes m)(L \otimes C^{-1})(C^{-1} \otimes L)(L \otimes C) \\
&= -(10\text{th term}).
\end{aligned}$$

We can similarly show that $1\text{st term} = -(11\text{th term})$, $-(2\text{nd term}) = 8\text{th term}$, $-(3\text{rd term}) = 5\text{th term}$, $-(7\text{th term}) = 9\text{th term}$. Consequently, (BJI) holds. We can similarly show that (BJI)' holds. \square

Readers can prove the above with the help of braiding diagrams.

Definition 1.4 Let $[\]$ be a morphism from $L \otimes L$ to L in \mathcal{C} . If (BJI) holds, then $(L, [\])$ is called a (left) Jacobi braided Lie algebra. If both of (BAS) and (BJI) hold then $(L, [\])$ is called a (left) strict Jacobi braided Lie algebra.

Dually, we can define right Jacobi braided Lie algebras and right strict Jacobi braided Lie algebras. Left (strict) Jacobi braided Lie algebras are called (strict) Jacobi braided Lie algebras in general.

By Theorem 1.3 we have

Corollary 1.5 *If $(L, [\])$ is a braided m -Lie algebra, then the following conditions are equivalent:*

- (i) $(L, [\])$ is a left (right) strict Jacobi braided Lie algebra.
- (ii) $mC_{L,L} = mC_{L,L}^{-1}$.
- (iii) $[\]C_{L,L} = [\]C_{L,L}^{-1}$.

Furthermore, if L is a space graded by G with bicharacter r , then the braided primitive Jacobi identity (BJI) becomes:

$$r(|\ c\ |, |\ a\ |)[a, [b, c]] + r(|\ b\ |, |\ a\ |)[b, [c, a]] + r(|\ c\ |, |\ b\ |)[c, [a, b]] = 0 \quad (*)$$

for any homogeneous elements $a, b, c \in L$. That is, L is a Jacobi braided Lie algebra if and only if $(*)$ holds. For convenience, we let $J(a, b, c)$ denote the left hand side of $(*)$.

We now recall the (left) braided Lie algebra defined by Majid [10, Definition 4.1]. A (left) braided Lie algebra in \mathcal{C} is a coalgebra (L, Δ, ϵ) in \mathcal{C} , equipped with a morphism $[\] : L \otimes L \rightarrow L$ satisfying the axioms:

- (L1) $([\])(L \otimes [\]) = [\]([\] \otimes [\])(L \otimes C \otimes L)(\Delta \otimes L \otimes L)$
- (L2) $C([\] \otimes L)(L \otimes C)(\Delta \otimes L) = (L \otimes [\])(\Delta \otimes L)$
- (L3) $[\]$ is a coalgebra morphism in (\mathcal{C}, C) .

Axiom (L1) is called the left braided Jacobi identity.

There exist braided m -Lie algebras which are neither braided Lie algebras nor Jacobi braided Lie algebras as is seen from the following example.

Example 1.6 (see [11]) *Let $L = F\{x\}/\langle x^n \rangle$ be an algebra in $({}^{F\mathbf{Z}_n}\mathcal{M}, C^r)$ with a primitive n th root q of 1 and $r(k, m) = q^{km}$ for any $k, m \in \mathbf{Z}_n$, where n is a natural number.*

(i) *If $n > 3$, then $(L, [\])$ is a braided m -Lie algebra but is neither a Jacobi braided Lie algebra nor a braided Lie algebra of Majid.*

(ii) *If $n = 3$, then $(L, [\])$ is a braided m -Lie algebra and a Jacobi braided Lie algebra but is neither a strict Jacobi braided Lie algebra nor a braided Lie algebra of Majid.*

Proof. (i) Since $J(x, x, x) = 3qx^3(1 - q - q^2 + q^3) \neq 0$ we have that $(L, [\])$ is not a Jacobi braided Lie algebra. If $(L, [\], \Delta, \epsilon)$ is a braided Lie algebra, since $[\]$ is a coalgebra homomorphism, we have that

$$\begin{aligned} [1\ 1] &= 0, \text{ implying } \epsilon(1) = 0 \\ [x, x] &= x^2(1 - q), \text{ implying } \epsilon(x^2)(1 - q) = \epsilon(x)^2 \\ [x, x^2] &= x^3(1 - q^2), \text{ implying } \epsilon(x^3)(1 - q^2) = \epsilon(x)\epsilon(x^2) \\ &\dots \\ [x, x^{n-1}] &= x^n(1 - q^{n-1}), \text{ implying } 0 = \epsilon(x^n)(1 - q^{n-1}) = \epsilon(x)\epsilon(x^{n-1}). \end{aligned}$$

Thus $\epsilon(x^m) = 0$ for $m = 0, 1, 2, \dots, n-1$, which contradicts the fact that (L, Δ, ϵ) is a coalgebra.

(ii) Since $J(x^k, x^l, x^m) = 0$ for any $k, l, m \in \mathbf{Z}_3$, we have that $(L, [\])$ is a Jacobi braided Lie algebra. It follows from $[x, x] = x^2(1 - q) \neq -q[x, x]$ that $(L, [\])$ is not a strict Jacobi braided Lie algebra. \square

Note that L in the above example may never be a braided Lie algebra in the vector space category ${}_F\mathcal{M}$ with the ordinary flip τ (i.e. $\tau(x \otimes y) = y \otimes x$) as braiding, although an extension of L may become a braided Lie algebra in ${}_F\mathcal{M}$. Furthermore, for the algebra $L = F\{x\}$ in $({}^F\mathbf{Z}\mathcal{M}, C^r)$ with $q^2 \neq 1$ and $r(k, m) = q^{km}$ for any $k, m \in \mathbf{Z}$, the above conclusion holds.

Definition 1.7 *Let $(L, [\])$ be a braided m -Lie algebra in (\mathcal{C}, C) . If M is an object and there exists a morphism $\alpha : L \otimes M \rightarrow M$ such that $\alpha([\] \otimes M) = \alpha(L \otimes \alpha) - \alpha(L \otimes \alpha)(C \otimes M)$, then (M, α) is called an L -module.*

2 Generalized Matrix Braided m -Lie Algebras

As examples of the braided m -Lie algebras, we introduce the concepts of generalized matrix algebras (see [14]) and generalized matrix braided m -Lie algebras. We construct generalized classical braided m -Lie algebras $sl_{q,f}(GM_G(A), F)$ and $osp_{q,t}(GM_G(A), M, F)$ of generalized matrix algebra $GM_G(A)$. We show how generalized matrix Lie color algebras are related to Lie superalgebras for any abelian group G . That is, we establish the relationship between generalized matrix Lie color algebras and Lie superalgebras.

Let I be a set. For any $i, j, l, k \in I$, we choose a vector space A_{ij} over field F and an F -linear map μ_{ijl} from $A_{ij} \otimes A_{jl}$ into A_{il} (written $\mu_{ijl}(x, y) = xy$) such that $x(yz) = (xy)z$ for any $x \in A_{ij}$, $y \in A_{jl}$, $z \in A_{lk}$. Let A be the external direct sum of $\{A_{ij} \mid i, j \in I\}$. We define the multiplication in A as

$$xy = \left\{ \sum_k x_{ik} y_{kj} \right\}$$

for any $x = \{x_{ij}\}, y = \{y_{ij}\} \in A$. It is easy to check that A is an algebra (possibly without unit). We call A a generalized matrix algebra, or a gm algebra in short, written as $A = \sum \{A_{ij} \mid i, j \in I\}$ or $GM_I(A)$. Every element in A is called a generalized matrix. We can easily define gm ideals and gm subalgebras. It is easy to define upper triangular generalized matrices, strictly upper triangular generalized matrices and diagonal generalized matrices under some total order \prec of I .

Proposition 2.1 *Let $A = \sum \{A_{ij} \mid i, j \in I\}$ be a gm algebra and G an abelian group with $G = I$. Then A is an algebra graded by G with $A_g = \sum_{i=j+g} A_{ij}$ for any $g \in G$. In this case, the gradation is called a generalized matrix gradation, or gm gradation in short.*

Proof. For any $g, h \in G$, see that

$$\begin{aligned} A_g A_h &= \left(\sum_{i=j+g} A_{ij} \right) \left(\sum_{s=t+h} A_{st} \right) \\ &\subseteq \sum_{i=t+h+g} A_{i,t+h} A_{t+h,t} \\ &\subseteq A_{g+h}. \end{aligned}$$

Thus $A = \sum \{A_{ij} \mid i, j \in I\} = \sum_{g \in G} A_g$ is a G -graded algebra. \square

If \mathcal{C} is a small preadditive category and $A_{ij} = \text{Hom}_{\mathcal{C}}(j, i)$ is a vector space over F for any $i, j \in I = \text{objects in } \mathcal{C}$, then we may easily show that $\sum \{A_{ij} \mid i, j \in I\}$ is a generalized matrix algebra. Furthermore, if $V = \bigoplus_{g \in G} V_g$ is a graded vector space over field F with $A_{gh} = \text{Hom}_F(V_h, V_g)$ for any $g, h \in G$, then we call braided m-Lie algebra $A = \sum \{A_{ij} \mid i, j \in G\}$ the general linear braided m-Lie algebra, written as $gl(\{V_g\}, F)$. If $\dim V_g = n_g < \infty$ for any $g \in G$, then $gl(\{V_g\}, F)$ is written as $gl(\{n_g\}, F)$. Its braided m-Lie subalgebras are called linear braided m-Lie algebras. In fact, $gl(\{n_g\}, F) = \{f \in \text{End}_F V \mid \ker f \text{ is finite codimensional}\}$. When G is finite, we may view $gl(\{n_g\}, F)$ as a block matrix algebra over F . When $G = 0$, we denote $gl(\{n_g\}, F)$ by $gl(n, F)$, which is the ordinary ungraded general linear Lie algebra.

Assume that D is a directed graph (D is possibly an infinite directed graph and also possibly not a simple graph). Let I denote the vertex set of D , x_{ij} an arrow from i to j and $x = (x_{i_1 i_2}, x_{i_2 i_3}, \dots, x_{i_{n-1} i_n})$ a path from i_1 to i_n via arrows $x_{i_1 i_2}, x_{i_2 i_3}, \dots, x_{i_{n-1} i_n}$. For two paths $x = (x_{i_1 i_2}, x_{i_2 i_3}, \dots, x_{i_{n-1} i_n})$ and $y = (y_{j_1 j_2}, y_{j_2 j_3}, \dots, y_{j_{m-1} j_m})$ of D with $i_n = j_1$, we define the multiplication of x and y as

$$xy = (x_{i_1 i_2}, x_{i_2 i_3}, \dots, x_{i_{n-1} i_n}, y_{j_1 j_2}, y_{j_2 j_3}, \dots, y_{j_{m-1} j_m}).$$

Let A_{ij} denote the vector space over field F with basis being all paths from i to j , where $i, j \in I$. Notice that we view every vertex i of D as a path from i to i , written e_{ii} and $e_{ii} x_{ij} = x_{ij} e_{jj} = x_{ij}$. We can naturally define a linear map from $A_{ij} \otimes A_{jk}$ to A_{ik} as $x \otimes y = xy$ for any two pathes $x \in A_{ij}, y \in A_{jk}$. We may easily show that $\sum \{A_{ij} \mid i, j \in I\}$ is a generalized matrix algebra, which is called a path algebra, written as $A(D)$ (see, [1, Chapter 3]).

Example 2.2 *There are finite-dimensional braided m-Lie algebras in braided tensor category $({}^{\mathbf{F}\mathbf{Z}_3} \mathcal{M}, C^r)$ with a primitive 3th root q of 1 and $r(k, m) = q^{km}$ for any $k, m \in \mathbf{Z}_3$. Indeed, for any natural number n , we can construct a generalized matrix braided m-Lie algebra $A = \sum \{A_{ij} \mid i, j \in \mathbf{Z}_3\}$ such that $\dim A = n$.*

- (i) *Let $A_{ij} = 0$ when $i \neq j$ but $A_{11} = F$. Thus $\dim A = 1$.*
- (ii) *Let $A_{ij} = 0$ when $i \neq j$ but $A_{11} = A_{22} = F$. Thus $\dim A = 2$.*

(iii) Let $A_{ij} = 0$ when $i \neq j$ but $A_{11} = A_{22} = A_{33} = F$. Thus $\dim A = 3$.

(iv) Let D be a directed graph with vertex set \mathbf{Z}_3 and only one arrow from 1 to 2. Set $A = A(D)$. It is clear $\dim(A_{ij}) = 0$ when $i \neq j$ but $\dim(A_{11}) = \dim(A_{22}) = \dim(A_{33}) = \dim(A_{12}) = 1$. Thus $\dim A = 4$.

(v) Let D be a directed graph with vertex set \mathbf{Z}_3 and only two arrows: one from 1 to 2 and other one from 1 to 3. Set $A = A(D)$. It is clear $\dim(A_{ij}) = 0$ but $\dim(A_{11}) = \dim(A_{22}) = \dim(A_{33}) = \dim(A_{12}) = \dim(A_{13}) = 1$. Thus $\dim A = 5$.

(vi) Let D be a directed graph with vertex set \mathbf{Z}_3 and only $n + 2$ arrows: one from 1 to 2, one from 2 to 3 and the others from 1 to 3. Set $A = A(D)$. It is clear $\dim(A_{ij}) = 0$ but $\dim(A_{11}) = \dim(A_{22}) = \dim(A_{33}) = \dim(A_{12}) = \dim(A_{23}) = 1$ and $\dim(A_{13}) = n + 1$. Thus $\dim A = n + 6$ for $n = 0, 1, \dots$.

Let A be a braided m -Lie algebra, G be an abelian group with a bicharacter r and W be a vector space over F .

Definition 2.3 If f is an F -linear map from $GM_G(A)$ to W and satisfies the following

$$(i) f(a) = \sum_{g \in G} r(g, g) f(a_{gg})$$

(ii) $f(a_{ij}b_{ji}) = f(b_{ji}a_{ij})$ for any $a, b \in A$ and $i, j \in G$, then f is called a generalized quantum trace function from gm algebra $GM_G(A)$ to W , written $tr_{q,f}$.

Set

$$sl_{q,f}(GM_G(A), F) = \{a \in GM_G(A) \mid tr_{q,f}(a) = 0\}.$$

By computation,

$$tr_{q,f}[a, b] = r(u, u)^{-1} r(g, g) \sum_{g \in G} (1 - r(u, u)^2 r(u, g) r(g, u)) tr_{q,f}(b_{g,g+u} a_{g+u,g})$$

for any homogeneous elements $a, b \in A, u \in G$. Thus we get

Lemma 2.4 $sl_{q,f}(GM_G(A), F)$ is a braided m -Lie algebra with $[A, A] \subseteq sl_{q,f}(GM_G(A), F)$ iff

$$\sum_{g \in G} (1 - r(u, u)^2 r(u, g) r(g, u)) tr_{q,f}(b_{g,g+u} a_{g+u,g}) = 0$$

for any homogeneous elements $a, b \in A, u \in G$ with $|a| = u$ and $|b| = -u$. If, in addition, $tr_{q,f}(A_{ij}A_{ji}) \neq 0$ for any $i, j \in G$, then $sl_{q,f}(GM_G(A), F)$ is a braided m -Lie algebra with $[A, A] \subseteq sl_{q,f}(GM_G(A), F)$ iff r is a skew symmetric bicharacter.

If $f(a_{gg}) = a_{gg}$ for any $g \in G$, then from definition 2.3 f is a generalized quantum trace from $GM_G(A)$ to $GM_G(A)$. If $GM_G(A) = M_n(F)$ is the full matrix algebra over F with $G = \mathbf{Z}_n$, $r(i, j) = 1$ and $f(a_{gg}) = a_{gg}$ for any $i, j, g \in \mathbf{Z}_n$, then f is the ordinary trace function. If $GM_G(A) = gl(\{n_g\}, F)$ and $f(a_{gg}) = tr(a_{gg})$ (i.e. $f(a_{gg})$ is the ordinary

trace of the matrix a_{gg}) for any $g \in G$, then f is the quantum trace of the graded matrix algebra $gl(\{n_g\}, F)$. In this case, $sl_{q,f}(gl(\{n_g\}, F))$ is simply written as $sl_q(\{n_g\}, F)$.

Let G be an abelian group with a bicharacter r . If t is an F -linear map from $GM_G(A)$ to $GM_G(A)$ such that $t(A_{ij}) \subseteq A_{ji}$, $(t(a))_{ij} = t(a_{ji})$ and $t(ab) = t(b)t(a)$ for any $i, j \in G$, $a, b \in GM_G(A)$, then t is called a generalized transpose on $GM_G(A)$. Given $0 \neq M \in GM_G(A)$, for any $u \in G$, let $osp_{q,t}(GM_G(A), M, F)_u = \{a \in GM_G(A)_u \mid t(a_{u+g,g})M_{u+g,h} = -r(g, u)M_{g,u+h}a_{u+h,h} \text{ for any } g, h \in G\}$ and $osp_{q,t}(GM_G(A), M, F) = \bigoplus_{u \in G} osp_{q,t}(GM_G(A), M, F)_u$.

Lemma 2.5 $osp_{q,t}(GM_G(A), M, F)$ is a braided m -Lie algebra iff

$$\sum_{g \in G} (1 - r(u, v)r(v, u))M_{g, h+u+v}a_{h+u+v, v+h}b_{v+h, h} = 0$$

for any homogeneous elements $a, b \in A$, $u, v \in G$ with $|a| = u$ and $|b| = v$. If, in addition, for any $i, j, k \in G$, $a_{ij} \neq 0$ implies $a_{ij}A_{jk} \neq 0$, then $osp_{q,t}(GM_G(A), M, F)$ is a braided m -Lie algebra iff r is a skew symmetric bicharacter.

Proof. Obviously, $osp_{q,t}(GM_G(A), M, F)_u$ is a subspace. It remains to check that $osp_{q,t}(GM_G(A), M, F)$ is closed under bracket operation. For any $a \in osp_{q,t}(GM_G(A), M, F)_u$, $b \in osp_{q,t}(GM_G(A), M, F)_v$ and $u, v, g, h \in G$, set $w = u + v$. See that

$$\begin{aligned} t([a, b]_{w+g,g})M_{w+g,h} &= t((ab - r(v, u)ba)_{w+g,g})M_{w+g,h} \\ &= t(b_{v+g,g})t(a_{w+g,g+v})M_{w+g,h} \\ &\quad - r(v, u)t(a_{u+g,g})t(b_{w+g,g+u})M_{w+g,h} \\ &= r(g + v, u)r(g, v)M_{g, h+w}b_{h+w, u+h}a_{u+h, h} \\ &\quad - r(v, u)r(g + u, v)r(g, u)M_{g, h+w}a_{h+w, v+h}b_{v+h, h}, \\ -r(g, w)M_{g, w+h}[a, b]_{w+h,h} &= -r(g, w)M_{g, w+h}a_{w+h, v+h}b_{v+h, h} \\ &\quad - r(g, w)r(v, u)M_{g, w+h}b_{w+h, u+h}a_{u+h, h}. \end{aligned}$$

Thus $[a, b] \in osp_{q,t}(GM_G(A), M, F)_w$ iff $(r(u, v)r(v, u) - 1)M_{g, w+h}a_{w+h, v+h}b_{v+h, h} = 0$ for any $g, h \in G$. \square

We now consider $sl_{q,f}(GM_G(A), F)$ and $osp_{q,t}(GM_G(A), M, F)$ when the bicharacter r is skew symmetric. In this case, they become Lie color algebras in $({}^F G \mathcal{M}, C^r)$, called special gm Lie color algebra and ortho-symplectic gm Lie color algebra, respectively.

It is well-known that a Lie color algebra $({}^F G \mathcal{M}, C^r)$ with finitely generated G is related to a Lie super algebra by [13, Theorem 2]. We now show how the above gm Lie color algebras are related to Lie superalgebras for any abelian group G .

Let G be an abelian group with a skew symmetric bicharacter r . Set $G_{\bar{0}} = \{g \in G \mid r(g, g) = 1\}$ and $G_{\bar{1}} = \{g \in G \mid r(g, g) = -1\}$. We define a new bicharacter: $r_0(g, h) = -1$

for $g, h \in G_{\bar{1}}$ and $r_0(g, h) = 1$ otherwise. It is clear that r_0 is a bicharacter too. Obviously, $(r_0)_0 = r_0$ for any skew symmetric bicharacter r on G . For convenience, let L^s denote the Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ for Lie color algebra L in $({}^F G \mathcal{M}, C^r)$, where $L_{\bar{0}} = \oplus_{i \in G_{\bar{0}}} L_i$ and $L_{\bar{1}} = \oplus_{i \in G_{\bar{1}}} L_i$ with $[x, y] = xy - r_0(|y|, |x|)yx$ for any $x \in L_g, y \in L_h, g, h \in G$ (see [13, Page 718]).

Let $A = \sum \{A_{ij} \mid i, j \in G\} = GM_G(A)$ be a generalized matrix algebra. Set $B_{\bar{i}, \bar{j}} = \sum_{g \in G_{\bar{i}}, h \in G_{\bar{j}}} A_{gh}$ for any $\bar{i}, \bar{j} \in \mathbf{Z}_2$ and $B = \sum \{B_{\bar{i}, \bar{j}} \mid \bar{i}, \bar{j} \in \mathbf{Z}_2\}$. We denote the generalized matrix algebra $GM_{\mathbf{Z}_2}(B)$ by $GM_{\mathbf{Z}_2}(A^s)$. For $a \in GM_G(A)$, if $b_{\bar{i}, \bar{j}} = \sum_{g \in G_{\bar{i}}, h \in G_{\bar{j}}} a_{gh}$ for any $\bar{i}, \bar{j} \in \mathbf{Z}_2$, then we denote the element b by a^s . When $G = \mathbf{Z}_2$ with $r(g, h) = (-1)^{gh}$ for any $g, h \in \mathbf{Z}_2$, we denote $tr_{q,f}$ by $tr_{s,f}$ and $osp_{q,t}$ by $osp_{s,t}$. We have

Theorem 2.6 (i) $sl_{q,f}(GM_G(A), F)^s = sl_{s,f}(GM_{\mathbf{Z}_2}(A^s), F)$.

(ii) $osp_{q,t}(GM_G(A), M, F)^s = osp_{s,t}(GM_{\mathbf{Z}_2}(A^s), M^s, F)$.

Proof. (i) For any $a \in GM_G(A)$, see that

$$\begin{aligned} tr_{q,f}(a) &= \sum_{g \in G_{\bar{0}}} r_0(g, g) tr_{q,f}(a_{gg}) + \sum_{g \in G_{\bar{1}}} r_0(g, g) tr_{q,f}(a_{gg}) \\ &= tr_{s,f}(a). \end{aligned}$$

This completes the proof of (i).

(ii) For any $a \in (osp_{q,t}(GM_G(A), M, F)^s)_{\bar{0}}$ with $a \in osp_{q,t}(GM_G(A), M, F)_u$ and $u \in G_{\bar{0}}$, let $\bar{i}, \bar{j} \in \mathbf{Z}_2$ and we see that

$$\begin{aligned} t(a_{\bar{i}, \bar{i}}) M_{\bar{i}, \bar{j}} &= \sum_{g, h \in G_{\bar{i}}, k \in G_{\bar{j}}} t(a_{gh}) M_{gk} \\ &= - \sum_{h \in G_{\bar{i}}, k \in G_{\bar{j}}} r_0(h, u) M_{h, k+u} a_{k+u, k} \\ &= -M_{\bar{i}, \bar{j}} a_{\bar{j}, \bar{j}}. \end{aligned}$$

This shows $a \in osp_{s,t}(GM_{\mathbf{Z}_2}(A^s), M^s, F)$. We can similarly prove the others. \square

In fact, the relations (i) and (ii) above define a G -grading of Lie superalgebras $sl_{s,f}(GM_{\mathbf{Z}_2}(A^s), F)$ and $osp_{s,t}(GM_{\mathbf{Z}_2}(A^s), M^s, F)$, respectively.

We may apply the above results to $gl(\{n_g\}, F)$ with the ordinary quantum trace $tr_q(a) = \sum_{g \in G} r(g, g) tr(a_{gg})$ and the ordinary transpose $t(a) = a'$. In this case, $sl_{q,f}$ and $osp_{q,f}$ are denoted by sl_q and osp_q , respectively. We have

Corollary 2.7 (i) $(sl_q(gl(\{V_g\}, F))^s = sl_s(V_{\bar{0}}, V_{\bar{1}}, F)$.

(ii) $(osp_q(gl(\{V_g\}, F))^s = osp_s(V_{\bar{0}}, V_{\bar{1}}, M^s, F)$.

3 Braided m-Lie Algebras in the Yetter-Drinfeld Category and Their Representations

In this section, we give another class of braided m-Lie algebras. We shall show that representations of an algebra A associated to a braided m-Lie algebra L are also representations of L . Furthermore, we show that if (M, ψ) is a faithful representation of L , then representations of $End_F M$ are also representations of L .

The category is the Yetter-Drinfeld category $({}^B_B\mathcal{YD}, C)$, where B is a finite dimensional Hopf algebra and C is a braiding with $C(x, y) = \sum (x_{(-1)} \cdot y) \otimes x_{(0)}$ for any $x \in M, y \in N$.

We use the Sweedler's notation for coproducts and comodules, i.e.

$$\Delta(a) = \sum_a a_1 \otimes a_2 \quad \text{and} \quad \psi(x) = \sum_x x_{(-1)} \otimes x_{(0)}$$

when $a \in H$ a coalgebra and $x \in M$ a left H -comodule.

Lemma 3.1 (see [15, Lemma 2.1 and Lemma 2.3 (iv)])

(i) If (V, α_V, ϕ_V) and (W, α_W, ϕ_W) are two Yetter-Drinfeld modules over B with $\dim B < \infty$, then $Hom_F(V, W)$ is a Yetter-Drinfeld module under the following module operation and comodule operation: $(b \cdot f)(x) = \sum b_1 \cdot f(S(b_2) \cdot x)$ and $\phi(f) = (S^{-1} \otimes \hat{\alpha})(b_B \otimes f)$, where $\hat{\alpha}$ is defined by $(b^* \cdot f)(x) = \langle b^*, x_{(-1)} S(f(x_{(0)}))_{(-1)} \rangle_{ev} (f(x_{(0)}))_{(0)}$ for any $x \in V, f \in Hom_F(V, W), b^* \in B^*$. Here b_B denotes a coevaluation and \langle, \rangle_{ev} an evaluation of B .

(ii) If (M, α_M, ϕ_M) is a Yetter-Drinfeld modules over B , $End_F M$ is an algebra in $({}^B_B\mathcal{YD}, C)$

Proof. (i) It is clear that $\sum f_{(-1)} f_{(0)}(x) = \sum (f(x_{(0)}))_{(-1)} S^{-1}(x_{(-1)}) \otimes (f(x_{(0)}))_{(0)}$ for any $x \in V, f \in Hom_F(V, W), b \in B$. Using this, we can show that $Hom_F(V, W)$ is a B -comodule. Similarly, we can show that $Hom_F(V, W)$ is a B -module. We now show

$$\sum (b \cdot f)_{(-1)} \otimes (b \cdot f)_{(0)} = \sum b_1 f_{(-1)} S(b_3) \otimes b_2 \cdot f_{(0)} \quad (1)$$

for any $f \in Hom_F(V, W), b \in B$. For any $x \in V$, see that

$$\begin{aligned} \sum (b \cdot f)_{(-1)} \otimes (b \cdot f)_{(0)}(x) &= \sum b_1 (f(S(b_4)) \cdot x_{(0)})_{(-1)} S(b_3) S^{-1}(x_{(-1)}) \\ &\quad \otimes b_2 \cdot (f(S(b_4)) \cdot x_{(0)})_{(0)}, \\ b_1 f_{(-1)} S(b_3) \otimes (b_2 \cdot f_{(0)})(x) &= b_1 f_{(-1)} S(b_4) \otimes b_2 \cdot f_{(0)}((S(b_3) \cdot x)) \\ &= \sum b_1 (f(S(b_4) \cdot x_{(0)}))_{(-1)} S(b_3) S^{-1}(x_{(-1)}) b_5 S(b_6) \\ &\quad \otimes b_2 \cdot (f(S(b_4) \cdot x_{(0)}))_{(0)} \\ &= \sum b_1 (f(S(b_4)) \cdot x_{(0)})_{(-1)} S(b_3) S^{-1}(x_{(-1)}) \\ &\quad \otimes b_2 \cdot (f(S(b_4)) \cdot x_{(0)})_{(0)}. \end{aligned}$$

Thus (1) holds and $Hom_F(V, W)$ is a Yetter-Drinfeld module.

(ii) Let $E = End_F M$ and m denote the multiplication of E . Now we show that m is a homomorphism of B -comodules. It is sufficient to show

$$\sum_{fg} (fg)_{(-1)} \otimes (fg)_{(0)} = \sum_{f,g} f_{(-1)} g_{(-1)} \otimes f_{(0)} g_{(0)}. \quad (2)$$

for any $f, g \in E$. Indeed, for any $x \in M$, see that

$$\begin{aligned} \sum_{fg} (fg)_{(-1)} \otimes (fg)_{(0)}(x) &= \sum_x (fg(x_0))_{(-1)} S^{-1}(x_{(-1)}) \otimes (fg(x_0))_{(0)}, \\ \sum_{f,g} f_{(-1)} g_{(-1)} \otimes f_{(0)} g_{(0)}(x) &= \sum_{f,x} f_{(-1)} (g(x_{(0)}))_{(-1)} S^{-1}(x_{(-1)}) \otimes f_{(0)} ((g(x_{(0)}))_{(0)}) \\ &= \sum_{f,x} (f((g(x_{(0)}))_{(0)}))_{(-1)} \\ &\quad S^{-1}((g(x_{(0)}))_{(0)})_{(-1)2} (g(x_{(0)}))_{(-1)1} S^{-1}(x_{(-1)}) \\ &\quad \otimes (f((g(x_{(0)}))_{(0)}))_{(0)} \\ &= (fg(x_0))_{(-1)} S^{-1}(x_{(-1)}) \otimes (fg(x_0))_{(0)}. \end{aligned}$$

Thus (2) holds. Similarly, we can show that m is a homomorphism of B -modules. \square

Example 3.2 Let (M, α_M, ϕ_M) be a Yetter-Drinfeld modules over B and L a subobject of $E = End_F M$. If L is closed under operation $[\] = m - mC_{L,L}$, then $(L, [\])$ is a braided m -Lie subalgebra of E^- .

By Lemma 3.1, we may define representations of braided m -Lie algebras in the Yetter-Drinfeld category.

Definition 3.3 Let $(L, [\])$ be a braided m -Lie algebra in $({}^B_B \mathcal{YD}, C)$. If M is an object in $({}^B_B \mathcal{YD}, C)$ with morphism $\psi : L \rightarrow End_F M$ such that ψ is a homomorphism of braided m -Lie algebras, i.e. $\psi[\] = m(\psi \otimes \psi) - m(\psi \otimes \psi)C_{L,L}$, then (M, ψ) is called a representation of $(L, [\])$.

Obviously, (M, ψ) is a representation of L iff (M, α) is an L -module (see definition 1.7), where the relation between two operations is $\alpha(a, x) = \psi(a)(x)$ for any $a \in L, x \in M$.

Proposition 3.4 Let $(L, [\])$ be an object in $({}^B_B \mathcal{YD}, C)$ with a morphism $[\] : L \otimes L \rightarrow L$. If M is an object in $({}^B_B \mathcal{YD}, C)$ with monomorphism $\psi : L \rightarrow End_F M$ such that $\psi[\] = m(\psi \otimes \psi) - m(\psi \otimes \psi)C_{L,L}$, then L is a braided m -Lie algebra and (M, ψ) is a faithful representation of $(L, [\])$.

Proof. By Lemma 3.1 $E = End_F M$ is an algebra in $({}^B_B \mathcal{YD}, C)$. By Definition 1.1, $(L, [\])$ is a braided m -Lie algebra. \square

Proposition 3.5 *Let $(L, [\])$ be a braided m -Lie algebra in $({}^B_B\mathcal{YD}, C)$ induced by multiplication of A through ϕ .*

- (i) If (M, ψ) is a representation of algebra A , then $(M, \psi\phi)$ is a representation of L .*
- (ii) If (M, ψ) is a representation of braided m -Lie algebra A^- , then $(M, \psi\phi)$ is a representation of L .*
- (iii) If (M, ψ) is a faithful representation of L and (N, φ) is a representation of algebra $End_F M$, then $(N, \varphi\psi)$ is a representation of L .*

Proof. (i) and (ii) follow from Definition 3.3.

(iii) By Lemma 3.1, $E = End_F M$ is an algebra in $({}^B_B\mathcal{YD}, C)$. Thus L is a braided m -Lie algebra induced by E through ψ . By Proposition 3.5, we complete the proof. \square

Let $(L, [\])$ be a braided m -Lie subalgebra of the path algebra $(F(D, \rho))^-$ with relations, then a representation (V, f) of D with $f_\sigma = 0$ for any $\sigma \in \rho$ is also a representation of L (see [1, Proposition II.1.7]).

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